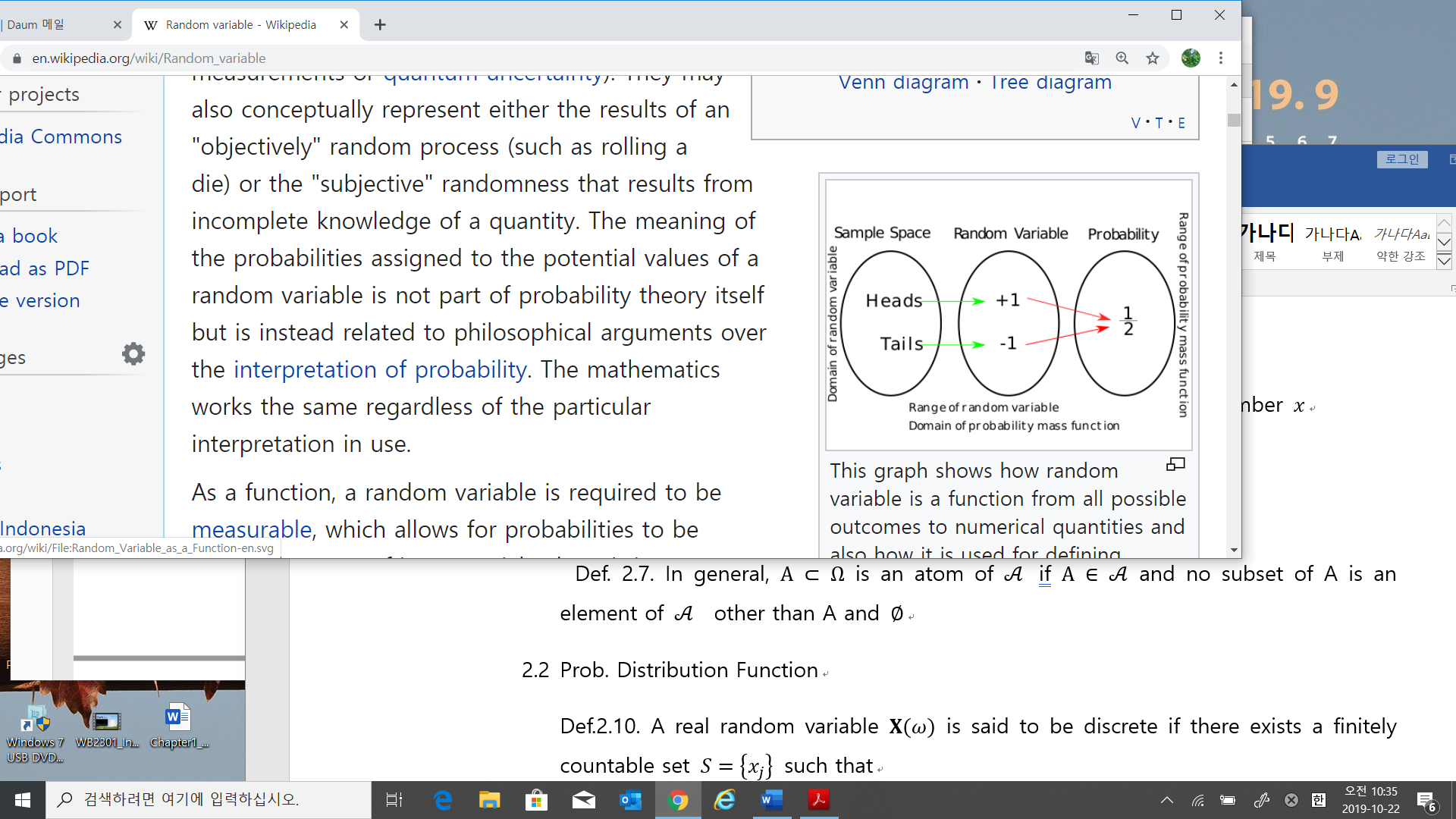
1. Random Variables and Stochastic Process
   1. Random Variables

Def. 2.1. Given a probability space,, a random variable is a real-(vector-) valued point function which carries a sample point, , into a point

in such a way that every sets, , of the form

is an element of the

* In the textbook, mis spelled. As
* Random variable
* is a function such that associates to a real number
* Wiki



Ex.2.4.

Given :

The Experiment: two coins flips

The sample space

The Event: at least one Head:

A candidate random variable

We may call this as the Indicator random variable

Now the event generated by are

Def. 2.7. In general, is an atom of if and no subset of A is an element of other than A and

* See the notation ,
  1. Prob. Distribution Function
* **Probability Distribution function**

Def.2.10. A real random variable is said to be discrete if there exists a finitely countable set such that

* 1. Prob. Density Function

Suppose such that

Then is the probability density function,

Proposition 2.12.

* Common (Probability) Distribution Functions for Random Variables

1. The uniform distribution function
2. The exponential distribution function
3. The Gaussian probability distribution(the normal distribution function)

A Gaussian random vector , the density is

where the mean vector,

the covariance matrix,

the determinant of

* 1. Probabilistic Concepts Applied to Random Variables
* Joint Probability Distribution
* Marginal Probability
* Joint Probability Density
* Marginal Probability Density
* The marginal probability distribution
* Ex.(Kim)

Given find the marginal

* Kim’s Example

1. Is it a CDF ?



1. Find
2. Is independent? No since

* Kim Example 2



Is this PDF independent? Yes…. Prove it.

Def 2.16. Two random variables and are called **independent** if any event of the form id independent of any event of the form where are sets in

* Fact
* The joint probability distribution
* The joint probability density function
* Marginal Probability Distribution

Let the joint PDF is

The marginal PDF of

can be calculated as

* 1. Functions of a Random Variable **-skip**

has the density function

Where stands for the absolute value of the determinant of the matrix

* 1. Expectations and Moments of a Random Variable

Def.

* The mean
* The sample mean

%% The same mean is a Random Variable! It is an estimator of the mean of a random variable . If is **independent identical distributed (iid)** random variable,i.e.,

Then the mean of the sample mean is

* Kim’s Comment

What is the difference between a) and b)? In order to use (b) , it is needed know the probability density function, whereas in (a), not needed.

Examp. 2.19. is uniformly distributed from 0 to 1,i.e.,

Then

Examp. 2.22 the expectation of the value of one roll of one die?

Properties

1. The operator of expectation is linear
2. The square mean / second moment
3. The higher order moment
4. The variance
5. The standard deviation
6. The sample variance

This is a random variable. And the **unbiased** estimator of

* Kim’s comment

What is the estimator? Let be a RV. I want to find a constant “C” as RV in some sense.

We may call C as a estimator of the RV . So there may be many estimator as you like.

We may classify the estimator as

1. Unbiased estimator / biased estimator

, then C is the unbiased estimator

1. The minimum variance estimator /the least square error estimator
2. The mean of is the minimum variance estimator / the least square error estimator.

Proof:

* , which minimizes the (c).

Examp. 2.24 The uniform distributed random variable

The Variance is

* 1. Characteristic Functions **-skip**

Lemma 2.27

Prop.2.28 If is a Gaussian random vector with mean, m, and covariance matrix P, then its characteristic function is

* Kim

Def : Two R.V. are **uncorrelated** if

Def: Two Gaussian R.Vectorsare uncorrelated if is a diagonal matrix

**Prop. 2.29. Uncorrelated Gaussian random variables are independent**

Theorem 2.30. If is a Gaussian random vector with mean , and covariance, , and if , where is a Gaussian random vector with zero mean and covariance, , then is a Gaussian random vector with mean, , and covariance, .

Theorem 2.30

**A R.V , another R.V. and they are independent**. Find mean and covariance of

* Characteristic function is difficult to remember. In the text book, using the characteristic method. In this case we may apply basic theory.

Sol: Let’s apply the basic definition.

Hence

* Theorem 2.30 is important. But **the assumption in the theorem is insufficient** as

**are independent.**

* In general, independency implies the uncorrelated, not vice versa
* However, in Gaussian Does satisfy the opposite direction.
* The covariance of a uncorrelated (so independent) Gaussian is a diagonal matrix,
* Linear matrix theory: similar transform

We know for any semi-positive symmetric matrix , there is a similar transform matrix such that

Hence the covariance for any gaussian Random vectors (correlated), there exits a such that

* Any Gaussian Random vectors, we can find a transformed Random Vectors which is uncorrelated (independent).
* Independency is important to calculate the probability. You know the Gaussian probability table, but it is a scalar. So it you want to calculate the joint probability which may be correlated, first find a similar transform matrix to generate a diagonal covariance matrix. Then you may calculate the joint probability as a separate probability.
* **The central limit theorem**

Theorem 2.31. Let be i.i.d. random variables with finite mean and variance,

and denote their sum as . Then the distribution of the normalized sum

is a Gaussian distribution with mean 0 and variance 1 in the limit as

* Proof : textbook P.52
* Remarks:

1. See, the condition, that means   
   the mean and the variance is constant, but the experiment is many time processing. For example,
2. A die, which is fair or not, you roll the same die many times. Then the mean of the sum () is a Gaussian if .
3. Some RV has no mean, then it will not be applicable.
   1. Conditional Expectations and Conditional Probabilities

* The conditional expectation
* Remarks
* is a constant, means it is not random variable.
* if is a constant, then is a constant
* if is a RV, then is a **Random Variable** of y
* **Iterated expectation** **(See the proof at p.57 and remember)**
* Kim’s comment

Even if we do not know .

* Kim Examp.

. , 🡪

Lemma 2.34.

* 1. Stochastic Process

Def. 2.36. A stochastic process is a family of random variables, , indexed by a real parameter and defined on a common probability space .

Ex. 2.37

Def. 2.38.

1. A stochastic process is said to be continuous in probability at t if

for all

1. A stochastic process is said to be separable if there exists a countable, dense set such that for any closed set

differ by a set such that

Theorem 2.40. The rational numbers in provide a separating set S.

Def. 2.42. Let X be a random process defined on the time interval, T. Let

be a partition of the time interval, T. If the increments, are mutually independent for any partition of T, then X is said to be a process with **independent increments**.

Def. 2.43 We say that a random process, X, is a Gaussian process if for every finite collection, the corresponding density function,

is a Gaussian density function.

Def. 2.44 We say that a random process X is a Gaussian process if every finite linear combination of the form

is a Gaussian random variable

Def 2.45. A random process, where T is a subset of the real line, is said to be a **Markov process** if for any increasing collection

or, equivalently

* 1. Gauss-Markov Processes – **The fundamental**

1. Dynamics

* State , is a known matrix, is a Gaussian Random sequence.

1. Given Conditions
2. Noise

where

1. The states

1. The correlation

which implies

1. The mean and covariance

* The mean
* The covariance
* The **transition** matrix – notation abuse
* We will discuss in the next Tuesday. However in discrete linear system(or Markov process) there is a definition of the transition matrix.
  1. Non-linear Stochastic Difference Equations 🡪 skip